DISCUSSIONS.

As the first discussion this month we present a paper which was read at the last annual meeting of the Mathematical Association of America, as part of a program devoted to the consideration of the sort of training in mathematics most useful for students specializing in fields in which mathematics finds frequent application. Professor Reed represents the point of view of the biometrist. His recommendations, briefly summarized, are: the usual courses in algebra, trigonometry, and analytic geometry; a short course in the calculus, emphasizing principles rather than technique; and a course in probability, with stress on statistical theory and the adjustment of curves to given data. It would seem, therefore, that the student of biometry will generally find ready at hand, in most of our colleges, courses agreeing reasonably well with the plan outlined by Professor Reed. Probably, too, the textbooks in elementary subjects are no longer so restricted in their treatment as Professor Reed implies. Few, if any, trigonometries of recent date, for example, fail to give applications to mechanics.

The brevity of the course in calculus in Professor Reed's scheme should be considered rather carefully. Is a student of statistics to accept Stirling's formula and the value of the probability integral on faith, or is he to receive demonstrations? The proofs, if given with logical precision, require more thorough treatment of the behavior of series and improper integrals than is usually found in even rather extensive first courses in calculus.

In the second discussion, Mr. Webb outlines a treatment of complex numbers, including the definition of exponential and trigonometric functions and their relationship. His scheme is similar to that found in some textbooks in trigonometry; seldom, if ever, is so full a discussion included in the ordinary course in algebra. Two items in his outline call for special comment. No. 6 prescribes "the customary development of e and of e^x by the binomial theorem as the limits of $(1 + 1/n)^n$ and $(1 + 1/n)^{nx}$ as $n \doteq \infty$." This customary scheme involves either a scandalously inaccurate treatment of the double limit process (as in most textbooks on calculus) or a degree of logical precision scarcely within the reach even of the average college graduate. The question involved is much more delicate than that of mere convergence. Either the properties of uniform convergence, or else some equivalent special process to avoid this concept must be used. It would be desirable, if possible, to find some less difficult approach to the exponential function. No. 8 implies that there must exist some k such that $\cos 1 + i \sin 1 = e^k$. Since the series for $\cos\theta$ and $\sin\theta$ are determined on the hypothesis that such a k exists, and then by use of the series the value $k = i$ is obtained, it is not easy to see how to avoid the hypothesis.

Professor Bell points out that the method of proof known as mathematical induction is valid only by virtue of a distinct assumption, which he formulates very clearly thus: *If a theorem is true for* $n = 1$, *and if its truth for* $n - 1$ *implies that it is true for* n, *then the theorem is true for all whole numbers.* This discussion should be helpful to our readers, inasmuch as the necessity of this explicit assumption is not always clearly recognized by teachers. However, the author's

implication that the rôle of such an assumption is overlooked by specialists in the logic of mathematics seems to be unwarranted. Writers on the foundations of mathematics generally recognize that the validity of mathematical induction is the result of a pertinent hypothesis. A common alternative to Professor Bell's form of statement is the following: *In any set of positive integers there is a least integer.* From this hypothesis, the other form of statement can be proved by considering the set of all numbers n (if any) for which the theorem under consideration is false.

Poincaré himself, in the sections of *Science and Hypothesis* criticized by Professor Bell, states that the principle of mathematical induction can not be obtained either from syllogisms based on the other fundamental axioms or from experience. For example[1]: "We may readily pass from one enunciation to another, and thus give ourselves the illusion of having proved that reasoning by recurrence is legitimate. But we shall always be brought to a full stop—we shall always come to an indemonstrable axiom, which will at bottom be but the proposition we had to prove translated into another language." Russell has long regarded mathematical induction as a *definition* of the *natural numbers*.[2] "We *define* the natural numbers as those to which proofs by mathematical induction can be applied." Since the number system of elementary arithmetic and algebra contains no other *integers* except these "natural numbers," it is in this special case only a matter of terminology to say which of the fundamental hypotheses shall be called *definitions* rather than merely *axioms*. Many logicians would prefer to regard the whole set of fundamental hypotheses underlying any mathematical system as collectively *defining* the system.

Professor DeCou gives an instance of a problem in maxima and minima arising in the printing office. The solution is simple; and in fact it may be shown by a slight alteration in notation that the problem belongs to the familiar type in which a sum of two variables is to be minimized while the product remains constant. It is published here on account of the value for the class room of problems arising from practical sources. How many similar instances may occur in trade, industry, and every day life, in which a little knowledge of mathematics would save, as here, "an added cost of \$100 to \$200," or an equivalent in energy or convenience?

One detail of the problem suggests a further interesting question. As the calculus refuses to discriminate between integers, fractions, and irrationals, the formal solution of the problem generally gives an irrational result for the number of electrotypes needed. Professor DeCou says, "Of course only the nearest integral value of x is used." It is fairly clear, since the relationship of C and x is represented graphically by a hyperbola, that the minimum C for integral x must be given by one of the two integers enclosing the value of x for which the actual minimum occurs. But it is not evident that of these two, the *nearest* will give the value desired. In fact, this turns out not to be precisely correct,

[1] *Science and Hypothesis*, Part I, Chapter VI.

[2] *Introduction to Mathematical Philosophy*, Chapter III.

and our readers may be interested in verifying the true result. If $\sqrt{PR/ES} = \alpha$, then α should be replaced by the next smaller integer n or the next greater integer $n + 1$, not according as α is less or greater than the arithmetic mean $n + \frac{1}{2}$, but according as α is less or greater than the geometric mean $\sqrt{n(n + 1)}$. For large values of n, the distinction between the two criteria is slight. Of course in any actual case it would be simple enough to substitute each of the two values in the expression for C in order to determine which was the better.

I. The Mathematics of Biometry.

By Lowell J. Reed, Johns Hopkins University.

(Read before the Mathematical Association of America, January 1, 1920.)

I think we are all agreed that the program for this meeting of the Mathematical Association of America is as important as any that the Association has ever discussed. Any science that fails to ally itself with the other sciences must of necessity have a narrow development, and this is perhaps more true of mathematics, due to its wide application, than of any of the other sciences. It is to be hoped that the Program Committee will carry out its own suggestion that at some later meeting we consider the converse of the present question, that is, the contribution of other sciences to the development of mathematics.

My own part in the present program is the discussion of the mathematics of biometry, and instead of presenting illustrations of the application of mathematics to this branch of biology I am going to outline what I consider to be the proper mathematical training for work in biometry. I wish first to call attention to the development of the present method of teaching mathematics. We have in the field of mathematics a number of branches that have been developed mainly as tools for the solution of practical problems. Thus, trigonometry has been developed mainly for the solution of problems in surveying; probability has been developed for games of chance. Now whenever this has been the case the teaching of the subject has been concentrated on this one particular application, to the exclusion of all others. In recent years the increasing use of mathematical methods in such sciences as chemistry, biology, etc., has led to an effort to broaden the teaching of mathematics by introducing new applications. The result has been a new group of textbooks under such titles as "Calculus for Chemical Students," "Mathematics for Agricultural Students," etc. All of these texts seem to me to miss the point in that they imply that calculus for chemistry students is distinct from calculus for other groups. This I do not believe to be the case. If we consider the mathematical needs of a student in any one of the sciences we find them about as follows: First, he needs a foundation in algebra, trigonometry, analytic geometry, and calculus; and secondly, he needs to be trained to take a problem in his particular field and translate it into mathematical language. The latter need is the greater of the two, and it is the one that it is the more difficult to satisfy.